

ON THE MAXIMAL OPERATORS OF WALSH-KACZMARZ-FEJÉR MEANS

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ABSTRACT. The main aim of this paper is to prove that the maximal operator $\sigma_p^{\kappa,*} f := \sup_{n \in \mathbf{P}} |\sigma_n^\kappa f| / (n+1)^{1/p-2}$ is bounded from the Hardy space H_p to the space L_p , for $0 < p < 1/2$.

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1. INTRODUCTION

The a.e. convergence of Walsh-Fejér means $\sigma_n f$ was proved by Fine [1]. In 1975 Schipp [11] showed that the maximal operator σ^* is of weak type $(1, 1)$ and of type (p, p) for $1 < p \leq \infty$. The boundedness fails to hold for $p = 1$. But, Fujii [2] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1 . The theorem of Fujii was extended by Weisz [24], he showed that the maximal operator σ^* is bounded from the martingale Hardy space H_p to the space L_p for $p > 1/2$. Simon gave a counterexample [13], which shows that the boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to Goginava [7] (see also [17]). In the endpoint case $p = 1/2$ two positive result was showed. Weisz [25] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$. In 2008 Goginava [8] (see also [18]) proved that the following is true:

TheoremA. The maximal operator $\tilde{\sigma}^*$ defined by

$$\tilde{\sigma}^* f := \sup_{n \in \mathbf{P}} \frac{|\sigma_n f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. He also proved that for any nondecreasing function $\varphi : \mathbf{P}_+ \rightarrow [1, \infty)$ satisfying the conditions

$$\lim_{n \rightarrow \infty} \frac{\log^2(n+1)}{\varphi(n)} = +\infty,$$

the maximal operator

$$\sup_{n \in \mathbf{P}} \frac{|\sigma_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

The author [19] proved that when $0 < p < 1/2$ the maximal operator

$$(1) \quad \sigma_p^* f := \sup_{n \in \mathbf{P}} \frac{|\sigma_n f|}{(n+1)^{1/p-2}}$$

with respect to Walsh system is bounded from the Hardy space H_p to the space L_p .

We also proved that for any nondecreasing function $\varphi : \mathbf{P}_+ \rightarrow [1, \infty)$ satisfying the conditions

$$(2) \quad \overline{\lim}_{n \rightarrow \infty} \frac{n^{1/p-2}}{\varphi(n)} = +\infty,$$

the maximal operator

$$\sup_{n \in \mathbf{P}} \frac{|\sigma_n f|}{\varphi(n)}$$

is not bounded from the Hardy space H_p to the space $L_{p,\infty}$ when $0 < p < 1/2$. Actually, we prove a stronger result than the unboundedness of the maximal operator $\tilde{\sigma}_p^{\kappa,*} f$ from the Hardy space H_p to the spaces $L_{p,\infty}$. In particular, we prove that

$$\sup_{n \in \mathbf{P}} \left\| \frac{\sigma_n f}{\varphi(n)} \right\|_{L_{p,\infty}} = \infty.$$

In 1948 Šneider [16] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n \rightarrow \infty} \frac{D_n^\kappa(x)}{\log n} \geq C > 0$$

holds a.e. In 1974 Schipp [12] and Young [21] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [15] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f . Gát [3] proved that, for any integrable functions, the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. He showed that the maximal operator of Walsh-Kaczmarz-Fejér means $\sigma^{\kappa,*}$ is weak type $(1, 1)$ and of type (p, p) for all $1 < p \leq \infty$. Gát's result was generalized by Simon [14], who showed that the maximal operator $\sigma^{\kappa,*}$ is of type (H_p, L_p) for $p > 1/2$. In the endpoint case $p = 1/2$ Goginava [6] proved that maximal operator $\sigma^{\kappa,*}$ is not of type $(H_{1/2}, L_{1/2})$ and Weisz [25] showed that the maximal operator is of weak type $(H_{1/2}, L_{1/2})$. Analogical results of Theorem A is proved in [5].

The main aim of this paper is to investigate (H_p, L_p) and $(H_p, L_{p,\infty})$ -type inequalities for the maximal operators $\sigma_p^{\kappa,*} f := \sup_{n \in \mathbf{P}} |\sigma_n^\kappa f| / (n+1)^{1/p-2}$ when $0 < p < 1/2$.

2. DEFINITIONS AND NOTATIONS

Now, we give a brief introduction to the theory of dyadic analysis [10]. Let \mathbf{P}_+ denote the set of positive integers, $\mathbf{P} := \mathbf{P}_+ \cup \{0\}$. Denote \mathbb{Z}_2 the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on \mathbb{Z}_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact groups \mathbb{Z}_2 . The elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$ ($k \in \mathbf{P}$). The group operation on G is the coordinatewise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) := G, \quad I_n(x) := I_n(x_0, \dots, x_{n-1}) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\},$$

$(x \in G, n \in \mathbf{P})$. These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbf{P}) \in G$ denote the null element of G , and $I_n := I_n(0)$ ($n \in \mathbf{P}$). Set $e_n := (0, \dots, 0, 1, 0, \dots) \in G$, the n -th coordinate of which is 1 and the rest are zeros ($n \in \mathbf{P}$).

For $k \in \mathbf{P}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the k -th Rademacher function. If $n \in \mathbf{P}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0, 1\}$ ($i \in \mathbf{P}$), i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{P} : n_j \neq 0\}$, that is $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k}, \quad (x \in G, n \in \mathbf{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

Skvortsov (see [15]) gave relation between the Walsh-Kaczmarz functions and Walsh-Paley functions by the of the transformation $\tau_A : G \rightarrow G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_1, x_0, x_A, x_{A+1}, \dots),$$

for $A \in \mathbf{P}$. By the definition we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_n(x)), \quad (n \in \mathbf{P}, x \in G).$$

The Dirichlet kernels are defined

$$D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k,$$

where $\alpha_n = w_n$ or κ_n ($n \in \mathbf{P}$), $D_0^\alpha := 0$. the 2^n -th Dirichlet kernels have a closed form (see e.g. [10])

$$(3) \quad D_{2^n}^w(x) = D_{2^n}^\kappa(x) = D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in I_n \\ 0 & \text{if } x \notin I_n \end{cases}$$

The norm (or quasinorm) of the space $L_p(G)$ is defined by

$$\|f\|_p := \left(\int_G |f(x)|^p d\mu(x) \right)^{1/p} \quad (0 < p < \infty).$$

The space $L_{p,\infty}(G)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\lambda > 0} \lambda \mu \{ |f| > \lambda \}^{1/p} \leq c < \infty.$$

The σ -algebra generated by the dyadic intervals of measure 2^{-k} will be denoted by F_k ($k \in \mathbf{P}$). Denote by $f = (f^{(n)}, n \in \mathbf{P})$ a martingale with respect to $(F_n, n \in \mathbf{P})$ (for details see, e. g. [22]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbf{P}} |f^{(n)}|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \in \mathbf{P}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right|, \quad x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$, then it is easy to show that the sequence $(S_{2^n}f : n \in \mathbf{P})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, \dots)$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\widehat{f}(i) = \lim_{k \rightarrow \infty} \int_G f^{(k)}(x) \alpha_i(x) d\mu(x), \quad (\alpha_i = w_i \text{ or } \kappa_i).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n}f : n \in \mathbf{P})$ obtained from f .

The partial sums of the Walsh-(Kaczmarz)-Fourier series are defined as follows:

$$S_M^\alpha(f; x) := \sum_{i=0}^{M-1} \widehat{f}(i) \alpha_i(x), \quad (\alpha = w \text{ or } \kappa).$$

For $n = 1, 2, \dots$ and a martingale f the Fejér means of order n of the Walsh-(Kaczmarz)-Fourier series of the function f is given by

$$\sigma_n^\alpha(f; x) = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\alpha(f; x), \quad (\alpha = w \text{ or } \kappa).$$

The Fejér kernel of order n of the Walsh-(Kaczmarz)-Fourier series defined by

$$K_n^\alpha(x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k^\alpha(x), \quad (\alpha = w \text{ or } \kappa).$$

For the martingale f we consider maximal operators

$$\begin{aligned}
\sigma^{\alpha,*} f &: = \sup_{n \in \mathbf{P}} |\sigma_n^\alpha f|, & (\alpha = w \text{ or } \kappa). \\
\tilde{\sigma}^{\alpha,*} f &: = \sup_{n \in \mathbf{P}} \frac{|\sigma_n^\alpha f|}{\log^2(n+1)}, & (\alpha = w \text{ or } \kappa). \\
\tilde{\sigma}_p^{\alpha,*} f &: = \sup_{n \in \mathbf{P}} \frac{|\sigma_n^\alpha f|}{(n+1)^{1/p-2}}, & (\alpha = w \text{ or } \kappa).
\end{aligned}$$

A bounded measurable function a is p -atom, if there exists a interval I , such that

$$\begin{cases} a) & \int_I a d\mu = 0, \\ b) & \|a\|_\infty \leq \mu(I)^{\frac{-1}{p}}, \\ c) & \text{supp}(a) \subset I. \end{cases}$$

3. FORMULATION OF MAIN RESULTS

Theorem 1. *Let $0 < p < 1/2$. Then the maximal operator $\tilde{\sigma}_p^{\kappa,*}$ is bounded from the Hardy space $H_p(G)$ to the space $L_p(G)$.*

Theorem 2. *Let $0 < p < 1/2$ and $\varphi : \mathbf{P}_+ \rightarrow [1, \infty)$ be a nondecreasing function satisfying the condition*

$$(4) \quad \lim_{n \rightarrow \infty} \frac{n^{1/p-2}}{\varphi(n)} = +\infty.$$

Then there exists a martingale $f \in H_p(G)$, such that

$$\sup_{n \in \mathbf{P}} \left\| \frac{\sigma_n f}{\varphi(n)} \right\|_{L_{p,\infty}(G)} = \infty.$$

4. AUXILIARY PROPOSITIONS

Lemma 1. (Weisz) [23] *Suppose that an operator T is sublinear and for some $0 < p \leq 1$*

$$\int_{\bar{I}} |Ta|^p d\mu \leq c_p < \infty$$

for every p -atom a , where I denote the support of the atom. If T is bounded from L_∞ to L_∞ , then

$$\|Tf\|_{L_p(G)} \leq c_p \|f\|_{H_p(G)}.$$

Lemma 2. (Gát) [3] *Let $A > t$, $t, A \in \mathbf{N}$, $z \in I_t \setminus I_{t+1}$. Then*

$$K_{2^A}^w(z) = \begin{cases} 0, & \text{if } z - z_t e_t \notin I_A, \\ 2^{t-1}, & \text{if } z - z_t e_t \in I_A. \end{cases}$$

If $z \in I_A$, then

$$K_{2^A}^w(z) = \frac{2^{A-1}}{2}.$$

Lemma 3. [4] *Let $n < 2^{A+1}$, $A > N$ and $x \in I_N(x_0, \dots, x_{m-1}, x_m = 1, 0, \dots, x_l = 1, 0, \dots, 0) =: J_N^{m,l}$, $l = 0, \dots, N-1$, $m = -1, \dots, l$. Then*

$$\int_{I_N} n |K_n^w(\tau_A(x+t))| d\mu(t) \leq \frac{c2^A}{2^{m+l}},$$

where $J_N^{-1,l} = I_N(0, \dots, x_l = 1, 0, \dots, 0)$.

Lemma 4. (Skvortsov) [15] *Let $n \in \mathbf{P}_+$. The following equality*

$$\begin{aligned} nK_n^\kappa(x) &= 1 + \sum_{i=0}^{|n|-1} 2^i D_{2^i}(x) + \sum_{i=0}^{|n|-1} 2^i r_i(x) K_{2^i}^w(\tau_i(x)) \\ &\quad + (n - 2^{|n|}) (D_{2^{|n|}}(x) + r_{|n|}(x) K_{n-2^{|n|}}^w(\tau_{|n|}(x))). \end{aligned}$$

holds.

Lemma 5. [9] *Let $2 < A \in \mathbf{P}$ and $q_A = 2^{2A} + 2^{2A-2} + \dots + 2^2 + 2^0$. Then*

$$q_{A-1} |K_{q_{A-1}}| \geq 2^{2m+2s-3},$$

for $x \in I_{2A}(0, \dots, 0, x_{2m} = 1, 0, \dots, 0, x_{2s} = 1, x_{2s+1}, \dots, x_{2A-1})$, $m = 0, 1, \dots, A-3$, $s = m+2$, $m+3, \dots, A-1$.

5. PROOF OF THE THEOREM

Proof of Theorem 1. Lemma 4 yields that

$$\begin{aligned} \tilde{\sigma}_n^\kappa f &= \frac{f * K_n^\kappa}{(n+1)^{1/p-2}} \\ &\leq \left| f * \frac{1}{(n+1)^{1/p-1}} \left(1 + \sum_{i=0}^{|n|-1} 2^i D_{2^i} \right) \right| \\ &\quad + \left| f * \frac{1}{(n+1)^{1/p-1}} \sum_{i=0}^{|n|-1} 2^i r_i K_{2^i}^w \circ \tau_i \right| \\ &\quad + \left| f * \frac{n - 2^{|n|}}{(n+1)^{1/p-1}} (D_{2^{|n|}} + r_{|n|} K_{n-2^{|n|}}^w \circ \tau_{|n|}) \right| \\ &= f * L_n^1 + f * L_n^2 + f * L_n^3 \end{aligned}$$

and

$$\tilde{\sigma}_n^{\kappa,*} f \leq \sum_{i=1}^3 \sup_{n \in P} |f * L_n^i| = \sum_{i=1}^3 R^i f.$$

The boundedness from L_∞ to L_∞ of the operators R^i follows (3) and

$$\|K_n^w \circ \tau_i\|_1 = \|K_n^w\|_1 \leq 2,$$

for $i < |n|$, $n \in \mathbf{P}$ (See [20]). By Lemma 1 the proof will be complete, if we show that the operators $R^i f$ are p -quasi-local. That is, there exists a constant c_p which depends only on p and

$$\int_{\bar{I}} |R^i a| d\mu \leq c_p < \infty, \quad i = 1, 2, 3,$$

for every p -atom a , where the dyadic interval I is the support of the p -atom a .

Let a be an arbitrary p -atom with support I , and $\mu(I) = 2^{-N}$. Without loss of generality, we may assume that $I := I_N$.

It is evident that $\tilde{\sigma}_n^\kappa(a) = 0$, if $n \leq 2^N$. Therefore, we can suppose that $n > 2^N$.

By $\|a\|_\infty \leq c2^{N/p}$ we have that

$$|a * L_n^i| \leq \int_{I_N} |a(s)| |L_n^i(x+s)| d\mu(s) \leq c2^{N/p} \int_{I_N} |L_n^i(x+s)| d\mu(s)$$

and

$$(5) \quad |R^i a| \leq c2^{N/p} \sup_{n > 2^N} \int_{I_N} |L_n^i(x+s)| d\mu(s).$$

Let $x \in I_j \setminus I_{j+1}$ for some $j = 0, \dots, N-1$ and $s \in I_N$, then $x+s \in I_j \setminus I_{j+1}$. Thus, we have

$$\begin{aligned} & \sup_{n > 2^N} \int_{I_N} |L_n^1(x+s)| d\mu(s) \\ & \leq \sup_{n > 2^N} \int_{I_N} \frac{1}{(n+1)^{1/p-1}} \left(1 + \sum_{i=0}^j 2^i D_{2^i}(x+s) \right) d\mu(s) \\ & \leq \frac{c}{2^{N(1/p-1)}} 2^{2j} 2^{-N} \leq \frac{c2^{2j}}{2^{N/p}}, \end{aligned}$$

Consequently,

$$\begin{aligned} & \int_{I_N} |R^1 a(x)|^p d\mu(x) \\ & = \sum_{j=0}^{N-1} \int_{I_j \setminus I_{j+1}} |R^1 a(x)|^p d\mu(x) \leq c \sum_{j=0}^{N-1} \int_{I_j \setminus I_{j+1}} 2^{2pj} d\mu(x) \\ & \leq c \sum_{j=0}^{N-1} 2^{(2p-1)j} \leq c < \infty, \quad 0 < p < 1/2. \end{aligned}$$

Now, we discuss $\int_{I_N} |R^2 a|^p d\mu$. Let $\overline{I_N} = \bigcup_{t=0}^{N-1} (I_t \setminus I_{t+1})$ and we decompose the sets $I_t \setminus I_{t+1}$ as the following disjoint union:

$$I_t \setminus I_{t+1} = \bigcup_{l=t+1}^N J_t^l,$$

where $J_t^l := I_N(0, \dots, 0, x_t = 1, 0, \dots, 0, x_l = 1, x_{l+1}, \dots, x_{N-1})$ for $t < l < N$ and $J_t^l := I_N(e_t)$ for $l = N$. Then we can write

$$\begin{aligned} \int_{I_N} |R^2 a(x)|^p d\mu(x) &= \sum_{t=0}^{N-1} \sum_{l=t+1}^N \int_{J_t^l} |R^2 a(x)|^p d\mu(x) \\ &= \sum_{t=0}^{N-1} \sum_{l=t+1}^{N-1} \int_{J_t^l} |R^2 a(x)|^p d\mu(x) + \sum_{t=0}^{N-1} \int_{J_t^N} |R^2 a(x)|^p d\mu(x) \\ &=: \sum_1 + \sum_2. \end{aligned}$$

From Lemma 2 we have

$$\begin{aligned} &\sup_{n > 2^N} \int_{I_N} |L_n^2(x + s)| d\mu(s) \\ &\leq \sup_{n > 2^N} \frac{1}{(n+1)^{1/p-1}} \int_{I_N} \sum_{i=0}^l 2^i |K_{2^i}^w(\tau_i(x + s))| d\mu(s) \\ &\leq \sup_{n > 2^N} \frac{c}{(n+1)^{1/p-1}} \int_{I_N} \left(\sum_{i=0}^t 2^{2i} + \sum_{i=t+1}^l 2^i 2^{i-t} \right) d\mu(s) \\ &\leq \frac{c(2^{2t} + 2^{2l-t})}{2^{N/p}}, \quad x \in J_t^l. \end{aligned}$$

Hence,

$$\begin{aligned} (6) \quad \sum_1 &\leq c \sum_{t=0}^{N-1} \sum_{l=t+1}^{N-1} \int_{J_t^l} (2^{2t} + 2^{2l-t})^p d\mu(x) \\ &\leq c \sum_{t=0}^{N-1} \sum_{l=t+1}^{\lfloor 3t/2 \rfloor} \int_{J_t^l} 2^{2pt} d\mu(x) + c \sum_{t=0}^{N-1} \sum_{l=\lfloor 3t/2 \rfloor + 1}^{N-1} \int_{J_t^l} 2^{p(2l-t)} d\mu(x) \\ &\leq c \sum_{t=0}^{N-1} \sum_{l=t+1}^{\lfloor 3t/2 \rfloor} 2^{2pt} 2^{-l} + c \sum_{t=0}^{N-1} \sum_{l=\lfloor 3t/2 \rfloor + 1}^{N-1} 2^{p(2l-t)} 2^{-l} < c < \infty. \end{aligned}$$

Let $x \in J_t^N$. Then Lemma 2 yields

$$\begin{aligned} &\sup_{n > 2^N} \int_{I_N} |L_n^2(x + s)| d\mu(s) \\ &\leq \sup_{n > 2^N} \frac{1}{(n+1)^{1/p-1}} \int_{I_N} \sum_{i=0}^{|n|-1} 2^i |K_{2^i}^w(\tau_i(x + s))| d\mu(s) \\ &\leq c \sup_{n > 2^N} \frac{1}{(n+1)^{1/p-1}} \left(\int_{I_N} \left(\sum_{i=0}^t 2^{2i} + \sum_{i=t+1}^N 2^i 2^{i-t} \right) d\mu(s) + \sum_{i=N+1}^{|n|-1} \int_{I_i(x_{N,i-1})} 2^i 2^{i-t} d\mu(s) \right) \\ &\leq c \sup_{n > 2^N} \frac{2^{2t-N} + 2^{N-t} + 2^{|n|-t}}{(n+1)^{1/p-1}} \leq c \left(\frac{2^{2t}}{2^{N/p}} + \frac{2^{2N}}{2^{N/p} 2^t} \right), \end{aligned}$$

where $x_{N,i-1} := \sum_{j=N}^{i-1} x_j e_j$.

Consequently,

$$\begin{aligned}
\sum_2 &\leq c \sum_{t=0}^{N-1} \int_{J_t^N} \left(2^{2t} + \frac{2^{2N}}{2^t}\right)^p d\mu(x) \\
&\leq c \sum_{t=0}^{[2N/3]} \int_{J_t^N} \frac{2^{2pN}}{2^{pt}} d\mu(x) + c \sum_{t=[2N/3]+1}^{N-1} \int_{J_t^N} 2^{2pt} d\mu(x) \\
&\leq c \sum_{t=0}^{[2N/3]} \frac{1}{2^{2pt}} + c \sum_{t=[2N/3]+1}^{N-1} \frac{2^{2pt}}{2^N} \leq c.
\end{aligned}$$

To discuss $\int_{\overline{I_N}} |R^3 a|^p d\mu$ we use Lemma 3 and the following disjoint decomposition of $\overline{I_N}$:

$$\overline{I_N} = \bigcup_{l=0}^{N-1} \bigcup_{m=-1}^l J_N^{l,m},$$

where the set $J_N^{l,m}$ is defined in Lemma 3.

If $x \in \overline{I_N}$ and $s \in I_N$, then $x + s \in \overline{I_N}$ and $D_{2^{|n|}}(x + s) = 0$, for $n > N$. Moreover, if $x \in J_N^{l,m}$, then $x + s \in J_N^{l,m}$ and by Lemma 3 we have

$$\begin{aligned}
&\sup_{n > 2^N} \int_{I_N} |L_n^3(x + s)| d\mu(s) \\
&\leq \sup_{n > 2^N} \int_{I_N} \frac{n - 2^{|n|}}{(n + 1)^{1/p-1}} |K_{n-2^{|n|}}^w(\tau_{|n|}(x + s))| d\mu(s) \\
&\leq c \sup_{n > 2^N} \frac{1}{(n + 1)^{1/p-1}} \frac{2^{|n|}}{2^{l+m}} \leq \frac{c 2^{2N}}{2^{N/p} 2^{l+m}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_{\overline{I_N}} |R^3 a(x)|^p d\mu(x) \\
&= \sum_{l=0}^{N-1} \sum_{m=-1}^l \int_{J_N^{l,m}} |R^3 a(x)|^p d\mu(x) \\
&\leq c \sum_{l=0}^{N-1} \sum_{m=-1}^l \int_{J_N^{l,m}} \frac{2^{2pN}}{2^{p(l+m)}} d\mu(x) \leq c \sum_{l=0}^{N-1} \sum_{m=-1}^l \frac{2^{2pN}}{2^{(l+m)p}} 2^{-N+m} \\
&\leq c \sum_{l=0}^{N-1} \sum_{m=-1}^l \frac{2^{(1-p)m}}{2^{pl} 2^{N(1-2p)}} \leq c \sum_{l=0}^{N-1} \frac{2^{(1-2p)l}}{2^{N(1-2p)}} < c < \infty.
\end{aligned}$$

Which complete the proof of Theorem 1.

Proof of Theorem 2. Let $\{m_k : k \in \mathbf{P}\}$ be an increasing sequence of positive integers such that

$$\lim_{k \rightarrow \infty} \frac{2^{2m_k(1/p-2)}}{\varphi(q_{m_k})} = +\infty.$$

Let

$$f_{m_k}(x) := D_{2^{2m_k+1}}(x) - D_{2^{2m_k}}(x).$$

It is evident that

$$\widehat{f}_{m_k}^\kappa(i) = \begin{cases} 1, & \text{if } i = 2^{2m_k}, \dots, 2^{2m_k+1} - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can write that

$$(7) \quad S_i^\kappa f_{m_k}(x) = \begin{cases} D_i^\kappa(x) - D_{2^{2m_k}}(x), & i = 2^{2m_k} + 1, \dots, 2^{2m_k+1} - 1, \\ f_{m_k}(x), & i \geq 2^{2m_k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Since,

$$f_{m_k}^*(x) = \sup_{n \in \mathbf{P}} |S_{2^n}(f_{m_k}; x)| = |f_{m_k}(x)|,$$

from (7) we get

$$(8) \quad \|f_{m_k}\|_{H_p} = \|f_{m_k}^*\|_p = \|D_{2^{2m_k}}\|_p = 2^{2m_k(1-1/p)}.$$

Since (see [15])

$$D_n^\kappa(x) = D_{2^{|n|}}(x) + r_{|n|}(x) D_{n-2^{|n|}}^w(\tau_{|n|}(x))$$

from (7) we can write

$$\begin{aligned} & \left| \frac{\sigma_{q_{m_k}}^\kappa f(x)}{\varphi(q_{m_k})} \right| \\ &= \frac{1}{\varphi(q_{m_k})} \frac{1}{q_{m_k}} \left| \sum_{j=0}^{q_{m_k}-1} S_j^\kappa f_{m_k}(x) \right| \\ &= \frac{1}{\varphi(q_{m_k})} \frac{1}{q_{m_k}} \left| \sum_{j=2^{2m_k}}^{q_{m_k}-1} S_j^\kappa f_{m_k}(x) \right| \\ &= \frac{1}{\varphi(q_{m_k})} \frac{1}{q_{m_k}} \left| \sum_{j=2^{2m_k}}^{q_{m_k}-1} (D_j^\kappa(x) - D_{2^{2m_k}}(x)) \right| \\ &= \frac{1}{\varphi(q_{m_k})} \frac{1}{q_{m_k}} \left| \sum_{j=0}^{q_{m_k}-1-1} (D_{j+2^{2m_k}}^\kappa(x) - D_{2^{2m_k}}(x)) \right| \\ &= \frac{1}{\varphi(q_{m_k})} \frac{1}{q_{m_k}} \left| \sum_{j=0}^{q_{m_k}-1-1} D_j^w(\tau_{2^{2m_k}}(x)) \right| \\ &= \frac{1}{\varphi(q_{m_k})} \frac{q_{m_k}-1}{q_{m_k}} \left| K_{q_{m_k}-1}^w(\tau_{2^{2m_k}}(x)) \right| \end{aligned}$$

Let $x \in J_{2^{2m_k}}^{2m_k-2s-1, 2m_k-2l-1}$ for some $l < s < m_k$. Then from Lemma 5 we have

$$\frac{|\sigma_{q_{m_k}}^\kappa f(x)|}{\varphi(q_{m_k})} \geq \frac{c 2^{2s+2l-2m_k}}{\varphi(q_{m_k})}.$$

Hence, we can write

$$\begin{aligned} & \mu \left\{ x \in G : \frac{|\sigma_{q_{m_k}}^\kappa f(x)|}{\varphi(q_{m_k})} \geq \frac{c}{2^{2m_k} \varphi(q_{m_k})} \right\} \\ & \geq \sum_{l=0}^{m_k-3} \sum_{s=l+2}^{m_k-1} \mu \left\{ J_{2m_k}^{2m_k-2s-1, 2m_k-2l-1} \right\} > c > 0. \end{aligned}$$

Then from (8) we obtain

$$\begin{aligned} & \frac{\frac{c}{2^{2m_k} \varphi(q_{m_k})} \left\{ \mu \left\{ x \in G : \frac{|\sigma_{q_{m_k}}^\kappa f_{m_k}(x)|}{\varphi(q_{m_k})} \geq \frac{c}{2^{2m_k} \varphi(q_{m_k})} \right\} \right\}^{1/p}}{\|f_{m_k}\|_{H_p}} \\ & \geq \frac{c}{2^{2m_k} \varphi(q_{m_k}) 2^{2m_k(1-1/p)}} \\ & = \frac{c 2^{2m_k(1/p-2)}}{(q_{m_k})} \rightarrow \infty, \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Theorem 2 is proved.

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